Supplementary Information for:
Bringing Entanglement to the High Temperature Limit

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We discuss the quantification of the entanglement and present the derivation of the propagating function used in Phys. Rev. Lett. 104—— (2010).

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ENTANGLEMENT QUANTIFICATION

Entanglement can be easily quantified for a bipartite system of continuous variables in a Gaussian state. The logarithmic negativity [1] gives a characterization of the amount of entanglement which can be distilled into singlets. In the case of Gaussian continuous variable states, only the covariance matrix is needed. The covariance matrix $\sigma$ is defined as

$$\sigma_{\xi_i \xi_j} = \frac{\langle \xi_i \xi_j + \xi_j \xi_i \rangle}{2} - \langle \xi_i \rangle \langle \xi_j \rangle$$

(1)

with $\xi_i = Q_1, Q_2, P_1, P_2$. The logarithmic negativity is defined as

$$E_N = -\frac{1}{2} \sum_{i=1}^{4} \log_2[\text{Min}(1, 2|l_i|)]$$

(2)

where $l_i$ are the symplectic eigenvalues of the covariance matrix. They are simply the normal eigenvalues of the matrix $-i \Sigma \sigma$, with $\Sigma$ the symplectic matrix

$$\sigma = \begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix}$$

(3)

and $1_2$ is the $2 \times 2$ identity matrix.

Whenever the logarithmic negativity of the system is zero, we have a separable state $\rho_s = \sum_i p_i \rho_1^{(i)} \otimes \rho_2^{(i)}$, and each oscillator can be described independently. In continuous variable systems, the amount of entanglement is unbounded from above, having as a limiting case the maximally entangled EPR wavefunction with $E_N \to \infty$.

DECOUPLING THE TOTAL SYSTEM IN NORMAL MODES

The dynamics of the dissipative coupled oscillators is studied using the Feynman and Vernon approach, for details we refer the interested reader to [2, 3] and omit here the technical details of the approach.

The Hamiltonian of the total system, $H = H_S + H_{SB} + H_B$, studied in [4] reads

$$H_S = \frac{p_1^2}{2m} + \frac{1}{2}m\omega^2 q_1^2 + \frac{p_2^2}{2m} + \frac{1}{2}m\omega^2 q_2^2 + c(t)q_1 q_2,$$

(4)

$$H_B + H_{SB} = \sum_{k=1}^{N} \frac{1}{2}m_k \omega_k^2 \left( x_k - \frac{c_k q_1}{m_k \omega_k^2} \right)^2 + \sum_{k=1}^{N} \frac{1}{2}m'_k \omega'_k^2 \left( x'_k - \frac{c'_k q_2}{m'_k \omega'_k^2} \right)^2,$$

(5)

where $c(t) = mc_0 + mc_1 \cos(\omega t)$. Introducing the normal modes coordinates $x_+$ and $x_-$ defined by

$$q_1 = \frac{1}{\sqrt{2}} (x_+ + x_-), \quad p_1 = \frac{1}{\sqrt{2}} (p_+ + p_-), \quad q_2 = \frac{1}{\sqrt{2}} (x_+ - x_-), \quad p_2 = \frac{1}{\sqrt{2}} (p_+ - p_-),$$

(6)
This expression suggests the introduction of new sets of coordinates, which implies that similar expressions stand for the momentum coordinates. After substituting \( H \) leaves for \( H_S \)

\[
H_S = \frac{p^2}{2m} + \frac{1}{2} m\Omega^2_\pm(t)x^2_\pm + \frac{p^2}{2m} + \frac{1}{2} m\Omega^2_\pm(t)x^2_\pm,
\]

where \( \Omega^2_\pm(t) = \omega^2 \pm c(t)/m \) and

\[
H_B + H_{SB} = \sum_{k=1}^{N} \left( \frac{1}{2m_k} p_k^2 + \frac{1}{2} m_k\omega_k^2 x_k^2 - \frac{1}{\sqrt{2}} c_k x_k (x_+ + x_-) - \sum_{m} \frac{c^2_k}{4m_k\omega_k^2} (x_+ + x_-)^2 \right)
\]

\[
+ \sum_{k=1}^{N} \frac{1}{2m'_k} p'_k^2 + \frac{1}{2} m'_k\omega_k'^2 x_k'^2 - \frac{1}{\sqrt{2}} c'_k x'_k (x_+ - x_-) + \frac{c^2_k}{4m'_k\omega_k'^2} (x_+ - x_-)^2.
\]

for \( H_B + H_{SB} \). These coordinates introduce a cross-term \( x_+ x_- \), which cancels out if

\[
\frac{c^2_k}{m_k\omega_k^2} = \frac{c^2_k}{m'_k\omega_k'^2}.
\]

This requirement does not mean that the oscillators in the baths are identical but their modes distributions. In the continuous limit, it implies that the spectral distributions characterizing the baths, \( J_1(\omega) \) and \( J_2(\omega) \), are the same. In this case,

\[
H_B + H_{SB} = \sum_{k=1}^{N} \left( \frac{1}{2m_k} p_k^2 + \frac{1}{2} m_k\omega_k^2 x_k^2 + \frac{1}{2m'_k} p'_k^2 + \frac{1}{2} m'_k\omega_k'^2 x_k'^2 - \frac{1}{\sqrt{2}} c_k x_k (x_+ + x_-) - \sum_{m} \frac{c^2_k}{4m_k\omega_k^2} (x_+ + x_-)^2 \right)
\]

\[
- \left( \frac{1}{\sqrt{2}} c_k x_k + \frac{1}{\sqrt{2}} c'_k x'_k \right) x_+ - \left( \frac{1}{\sqrt{2}} c_k x_k - \frac{1}{\sqrt{2}} c'_k x'_k \right) x_-. \tag{10}
\]

This expression suggests the introduction of new sets of coordinates \( q_k \) and \( \Omega_k \) defined by

\[
q_k = \frac{1}{\Lambda_k \sqrt{2}} \left( c_k x_k + c'_k x'_k \right), \quad \Omega_k = \frac{1}{\Lambda_k \sqrt{2}} \left( c_k x_k - c'_k x'_k \right),
\]

which can be inverted

\[
x_k = \frac{1}{\sqrt{2} c_k} \left( \lambda_k q_k + \Lambda_k \Omega_k \right), \quad x'_k = \frac{1}{\sqrt{2} c'_k} \left( \lambda_k q_k - \Lambda_k \Omega_k \right),
\]

similar expressions stand for the momentum coordinates. After substituting (11) in (10) for \( H_B + H_{SB} \) and choosing \( \lambda_k = c_k, \Lambda_k = c'_k \) and \( m_k \omega_k^2 = m'_k \omega_k'^2 \) to eliminate terms proportional to \( p_k q_k, q_k \Omega_k, x_+ \Omega_k \) and \( x_- q_k \), we get

\[
H_B + H_{SB} = \sum_{k=1}^{N} \left\{ \frac{1}{2m_k} p_k^2 + \frac{1}{2} m_k\omega_k^2 \left( q_k - \frac{c_k}{m_k \omega_k^2} x_+ \right)^2 + \frac{1}{2m'_k} p'_k^2 + \frac{1}{2} m'_k\omega_k'^2 \left( \Omega_k - \frac{c'_k}{m'_k \omega_k'^2} x_- \right)^2 \right\}
\]

and therefore

\[
H = \frac{p^2}{2m} + \frac{1}{2} m\Omega^2_+ x^2_+ + \frac{1}{2} m\Omega^2_- x^2_- + \sum_{k=1}^{N} \left\{ \frac{1}{2m_k} p_k^2 + \frac{1}{2} m_k\omega_k^2 \left( q_k - \frac{c_k}{m_k \omega_k^2} x_+ \right)^2 \right\}
\]

\[
+ \frac{p^2}{2m} + \frac{1}{2} m\Omega^2_- x^2_- + \sum_{k=1}^{N} \left\{ \frac{1}{2m'_k} p'_k^2 + \frac{1}{2} m'_k\omega_k'^2 \left( \Omega_k - \frac{c'_k}{m'_k \omega_k'^2} x_- \right)^2 \right\}.
\]

In the following, we shall assume that the baths are initially at thermal equilibrium and choose as their initial state the thermal one; however, since our objective is to decouple completely the time evolution of the normal modes, we have to check that the product by pairs of the equilibrium density-matrix of the baths modes remains uncorrelated in the new coordinates. This implies that \( c_k = c'_k \), \( m_k = m'_k \) and \( \omega_k = \omega'_k \). So, in this microscopic description, the normal modes are coupled to identical but independent baths. It worths mentioning that not only the baths have the same modes, \( \frac{c^2_k}{m_k \omega_k^2} = \frac{c^2_k}{m'_k \omega_k'^2} \), but also the coupling between the system and the bath is the same, \( c_k = c'_k \). So, it reduces our baths to be equal in detail, we mean, oscillator by oscillator. Only at this point we can affirm that the normal modes will evolve independently. Note that the normal modes are coupled to the bath in different coordinates than the real modes; however, the introduction of the normal modes for the bath leaves the Jacobian of the transformation equals to 1, so after tracing the result will not be affected.
PROPAGATING FUNCTION FOR THE DENSITY MATRIX

In normal-modes coordinates, the evolution of the density matrix is governed by,

\[
\rho(x_+, y_+, x_-, y_-, t) = \int dx_+ dy_+ dx_- dy_- J(x_+, y_+, x_-, y_-; t|x_+, y_+, x_-, y_-, 0) \\
\times \rho(x_+, y_+, x_-, y_-, t),
\]

where \( J(x_+, y_+, x_-, y_-, t|x_+, y_+, x_-, y_-, 0) \) is the propagating function of the reduced density matrix,

\[
J(x_+, y_+, x_-, y_-, t|x_+, y_+, x_-, y_-, 0) = \int dx_+ \int dy_+ \int dx_- \int dy_- \\
\times \exp \left\{ \frac{i}{\hbar} S[x_+, x_-] - S[y_+, y_-] \right\} F[x_+, y_+, x_-, y_-],
\]

where \( S[x_+, x_-] \) is the classical action and \( F[x_+, y_+, x_-, y_-] \) the influence functional. \( \mathcal{D}x \) denotes an infinite product of measures in configuration space and implies a path integration over the paths \( x_+(t), y_+(t), x_-(t) \) and \( y_-(t) \) with endpoints \( x_+(0) = x_+, y(0) = y_+, x_-(0) = x_-; y(0) = y_-; x_+(t) = x_+, y(t) = y_+, x_-(t) = x_-; y(t) = y_- \). However, at this point we have decoupled our system and we are describing it by two different harmonic oscillators coupled to identical but independent baths. So,

\[
\rho(x_+, y_+, x_-, y_-, t) = \int dx_+ dy_+ dx_- dy_- J_+(x_+, y_+, x_-, y_-, t|x_+, y_+, x_-, y_-, 0) J_-(x_-, y_-, x_+, y_+, t|x_-, y_-, x_+, y_+, 0) \\
\times \rho(x_+, y_+, x_-, y_-, t),
\]

with

\[
J_{\pm}(x_+, y_+, t|x_+, y_+, t|x_+, y_+, 0) = \int dx_+ \int dy_+ \exp \left\{ \frac{i}{\hbar} S_{\pm}[x_+] - S_{\pm}[y_+] \right\} F[x_+, y_+].
\]

For the case of a bath modelled by harmonic oscillators \([5]\), the general result for \( F[x_\pm, y_\pm] \) was derived by Caldeira and Leggett \([2]\) and it reads

\[
F[x_\pm, y_\pm] = \exp \left\{ -\frac{i}{\hbar} m \left[ (x_{\pm, i} + y_{\pm, i}) \int_0^t ds \gamma(s)[x_\pm(s) - y_\pm(s)] + \int_0^t ds \int_0^s du \gamma(s-u)[\dot{x}_\pm(u) + \dot{y}_\pm(u)](x_\pm(s) - y_\pm(s)) \right] \right\} \\
\times \exp \left\{ -\frac{1}{\hbar} \int_0^t ds \int_0^s du [x_\pm(u) - y_\pm(u)] K(u - s)[x_\pm(s) - y_\pm(s)] \right\}.
\]

\( K(s) \) denotes the noise kernel

\[
K(s) = \int_0^\infty \frac{d\omega}{\omega} \coth \left( \frac{\omega \hbar}{2k_B T} \right) \cos(\omega s) I(\omega),
\]

wherein \( k_B \) denotes the Boltzmann constant and \( T \) the temperature of the bath. The friction kernel \( \gamma(s) \) in terms of the spectral density reads

\[
\gamma(s) = \frac{2}{m} \int_0^\infty \frac{d\omega}{\pi} \frac{I(\omega)}{\omega} \cos(\omega s), \quad \text{for Ohmic dissipation } \gamma(s) = 2\gamma_0(s).
\]

Since path integrals in \( J \) are quadratic, they can be done exactly to yield

\[
J = \frac{1}{N_+(t)N_-(t)} \exp \left\{ \frac{i}{\hbar} \left( S_+[x_+^f] - S_+[y_+^f] + S_-[x_-^f] - S_-[y_-^f] \right) \right\} F[x_+^f, y_+^f] F[x_-^f, y_-^f],
\]

being \( N_\pm \) a normalization factor determined by the normalization of the propagator. To simplify further expressions, let’s us to introduce the half-sum, \( q_\pm \), and difference, \( Q_\pm \), coordinates

\[
q_\pm = x_\pm - y_\pm, \quad Q_\pm = \frac{1}{2}(x_\pm + y_\pm).
\]
where \( N \) is given by the last definitions. We can express the master equation for a single driven harmonic oscillator as

\[
\dot{q}_{\pm}(s) - \gamma \dot{q}_{\pm}(s) + \Omega_{\pm}^2(s)q_{\pm}(s) = -2q_{f,\pm}\gamma \delta(t-s),
\]

\[
\dot{Q}_{\pm}(s) + \gamma \dot{Q}_{\pm}(s) + \Omega_{\pm}^2(s)Q_{\pm}(s) = -2Q_{f,\pm}\gamma \delta(s).
\]

It is important to mention that the solution to these equations will be valid only for \( s > 0 \) and it reads \[6\]

\[
q_{\pm}(s) = v_{1,\pm}(t,s)q_{i,\pm} + v_{2,\pm}(t,s)q_{f,\pm}, \quad Q_{\pm}(s) = u_{1,\pm}(t,s)Q_{i,\pm} + u_{2,\pm}(t,s)Q_{f,\pm}.
\]

Since baths are defined by the same spectral density, then note that \( \gamma \) is the same for \( \pm \) cases. So we have that

\[
J(x_+, y_+, x_-, y_-, t|x_+, y_+, x_-, y_-, 0) = \frac{1}{N(t)}
\]

\[
\times \exp \left[ -\frac{i}{\hbar}m \left\{ [b_{3,+}(t)q_{+,i} - b_{1,+}(t)q_{+,f}]Q_{+,f} + [b_{1,-}(t)q_{-,i} - b_{2,-}(t)q_{-,f}]Q_{-,f} \right\} \right]
\]

\[
\times \exp \left[ -\frac{i}{\hbar}m \left\{ [b_{3,-}(t)q_{-,i} - b_{1,-}(t)q_{-,f}]Q_{+,f} + [b_{1,+}(t)q_{+,i} - b_{2,+}(t)q_{+,f}]Q_{-,f} \right\} \right]
\]

\[
\times \exp \left[ -\frac{1}{\hbar}m \left( a_{11,+}(t)q_{+,i}^2 + a_{12,+}(t)q_{+,i}q_{+,f} + a_{22,+}(t)q_{+,f}^2 \right) \right]
\]

\[
\times \exp \left[ -\frac{1}{\hbar}m \left( a_{11,-}(t)q_{-,i}^2 + a_{12,-}(t)q_{-,i}q_{-,f} + a_{22,-}(t)q_{-,f}^2 \right) \right],
\]

where \( N(t) = N_+(t)N_-(t) \).

\[
a_{ij,\pm} = \frac{1}{2} \int_0^t ds_1 \int_0^t ds_2 v_{i,\pm}(t, s_1)v_{j,\pm}(t, s_2)K(s_1 - s_2),
\]

and

\[
b_{1,\pm}(t) = \dot{u}_{1,\pm}(t, 0) + \gamma, \quad b_{2,\pm}(t) = \dot{u}_{2,\pm}(t, 0), \quad b_{3,\pm}(t) = \dot{u}_{3,\pm}(t, 0), \quad b_{4,\pm} = \dot{u}_{4,\pm}(t, t),
\]

Using last definitions we can express \( N_{\pm} \) as \( N_{\pm} = \frac{2\pi \hbar}{m \text{Im}(\gamma)} \). Form Eq. (26) it is possible to derive and associated master equation. We based our calculation on the paper of Zerbe and Hänggi \[6\] where they derived the exact quantum master equation for a single driven harmonic oscillator.

**QUANTUM MASTER EQUATION (QME)**

The quantum master equation for the normal modes of the initial system reads

\[
i\hbar \frac{\partial}{\partial t} \rho(x_+, y_+, x_-, y_-) = \left[ -\frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial x_+^2} - \frac{\partial^2}{\partial y_+^2} \right) + \frac{m}{2} \Omega_+^2(t)(x_+^2 - y_+^2) \right] \rho(x_+, y_+, x_-, y_-)
\]

\[
+ \left[ -\frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial x_-^2} - \frac{\partial^2}{\partial y_-^2} \right) + \frac{m}{2} \Omega_-^2(t)(x_-^2 - y_-^2) \right] \rho(x_+, y_+, x_-, y_-)
\]

\[
- \frac{i\hbar\gamma}{2} (x_+ - y_+) \left( \frac{\partial}{\partial x_+} + \frac{\partial}{\partial y_+} \right) \rho(x_+, y_+, x_-, y_-) + iD_{+pp}(t, 0)(x_+^2 - y_+^2)\rho(x_+, y_+, x_-, y_-)
\]

\[
- \frac{i\hbar\gamma}{2} (x_- - y_-) \left( \frac{\partial}{\partial x_-} + \frac{\partial}{\partial y_-} \right) \rho(x_+, y_+, x_-, y_-) + iD_{-pp}(t, 0)(x_-^2 - y_-^2)\rho(x_+, y_+, x_-, y_-)
\]

\[
- \frac{\hbar}{m} [D_{+xp}(t, 0) + D_{+qw}(t, 0)](x_+ - y_+) \left( \frac{\partial}{\partial x_+} + \frac{\partial}{\partial y_+} \right) \rho(x_+, y_+, x_-, y_-)
\]

\[
- \frac{\hbar}{m} [D_{-xp}(t, 0) + D_{-qw}(t, 0)](x_- - y_-) \left( \frac{\partial}{\partial x_-} + \frac{\partial}{\partial y_-} \right) \rho(x_+, y_+, x_-, y_-),
\]
where

$$D_{\pm, pp}(t, 0) = 2 \left( b_{1, \pm} + \frac{\dot{b}_{2, \pm}}{b_{2, \pm}} \right) a_{22, \pm} - a_{22, \pm} + 2 \frac{\dot{b}_{2, \pm} b_{4, \pm}}{b_{2, \pm} b_{3, \pm}} - \frac{b_{4, \pm}}{b_{3, \pm}} \dot{a}_{12, \pm},$$

$$D_{\pm, px}(t, 0) = D_{\pm, xp}(t, 0) = -\frac{1}{b_{3, \pm}} \dot{a}_{12, \pm} + a_{22, \pm} + \frac{\dot{b}_{2, \pm}}{b_{2, \pm} b_{3, \pm}} a_{12, \pm}.$$  

For small values of $h$, $D_{\pm, px}(t, 0)$ and $D_{\pm, pp}(t, 0)$ can be written as [7]

$$D_{\pm, pp}(t, 0) = \frac{m \gamma}{\beta} + 2 \frac{m^2 \gamma A}{\beta} \left( \Omega_+^2(t) - \gamma^2 \right), \quad D_{\pm, px}(t, 0) = \frac{2 m \gamma^2 A}{\beta},$$

where $\Lambda = h^2 \beta^2 / 24m$.

**MEAN VALUES AND VARIANCES**

The expectation value $\langle f(x_{\pm}) \rangle$ of an observable $f$ is given by

$$\langle f(x_{\pm}) \rangle = \int dQ \, f(Q) \, \rho(Q, x_{\pm}, 0, t).$$

In terms of the initial values $\langle x_{\pm}(t_0) \rangle = \langle (x_{\pm, 0}) \rangle$ and $\langle p_{\pm}(t_0) \rangle = \langle (p_{\pm, 0}) \rangle$, the first moments read

$$\langle x_{\pm}(t) \rangle = \langle f_{2, \pm}(t) - \gamma \frac{\gamma}{2} f_{1, \pm}(t) \rangle \langle (x_{\pm, 0}) \rangle + \frac{1}{m} f_{1, \pm}(t) \langle (p_{\pm, 0}) \rangle,$$

$$\langle p_{\pm}(t) \rangle = m \frac{d}{dt} \langle x_{\pm}(t) \rangle = m \langle f_{2, \pm}(t) - \gamma \frac{\gamma}{2} f_{1, \pm}(t) \rangle \langle (x_{\pm, 0}) \rangle + \frac{1}{m} f_{1, \pm}(t) \langle (p_{\pm, 0}) \rangle.$$ 

The evolution of $\langle p_{\pm}(t) \rangle$ is discontinuous at $t_0 = 0$, i.e., $\lim_{t \to 0^+} \langle p_{\pm}(t) \rangle = \langle (p_{\pm, 0}) \rangle - m \gamma \langle x_{\pm, 0} \rangle / 2$ is in general not equal to $\langle p_{\pm, 0} \rangle$ [6]. This instantaneous jump of $\langle p_{\pm}(t) \rangle$ can be removed with an environmental cutoff $\omega_c$, or a non-factorizing initial state [3]. In this way, the variances, $\sigma_{x_{\pm} x_{\pm}}(t) = \langle x_{\pm}^2 \rangle - \langle x_{\pm} \rangle^2$, $\sigma_{x_{\pm} p_{\pm}}(t) = \frac{1}{2} \langle (x_{\pm} + p_{\pm})^2 \rangle - \langle x_{\pm} \rangle \langle p_{\pm} \rangle$ and $\sigma_{p_{\pm} p_{\pm}}(t) = \langle p_{\pm}^2 \rangle - \langle p_{\pm} \rangle^2$, are given by

$$\sigma_{x_{\pm} x_{\pm}}(t) = \left( f_{2, \pm} - \gamma \frac{\gamma}{2} f_{1, \pm} \right)^2 \sigma_{x_{\pm} x_{\pm}}^0 + \frac{2}{m} f_{1, \pm} \left( f_{2, \pm} - \gamma \frac{\gamma}{2} f_{1, \pm} \right) \sigma_{x_{\pm} p_{\pm}}^0 + \frac{1}{m^2} f_{1, \pm}^2 \sigma_{p_{\pm} p_{\pm}}^0 + \frac{2 h}{m} f_{1, \pm}^2 a_{11, \pm},$$

$$\sigma_{x_{\pm} p_{\pm}}(t) = m \left[ f_{2, \pm} - \frac{\gamma}{2} f_{1, \pm} - f_{1, \pm} \right] \sigma_{x_{\pm} x_{\pm}}^0 + \frac{1}{m^2} f_{1, \pm}^2 \sigma_{p_{\pm} p_{\pm}}^0 + \frac{2 h}{m} f_{1, \pm} a_{11, \pm} + f_{1, \pm} a_{12, \pm},$$

$$\sigma_{p_{\pm} p_{\pm}}(t) = m^2 \left( f_{2, \pm} - \gamma \frac{\gamma}{2} f_{1, \pm} \right)^2 \sigma_{x_{\pm} x_{\pm}}^0 + 2 m f_{1, \pm} \left( f_{2, \pm} - \gamma \frac{\gamma}{2} f_{1, \pm} \right) \sigma_{x_{\pm} p_{\pm}}^0 + f_{1, \pm}^2 \sigma_{p_{\pm} p_{\pm}}^0 + 2 h f_{1, \pm} a_{12, \pm} + 2 h m f_{1, \pm} a_{11, \pm}.$$

where $\sigma_{x_{\pm} x_{\pm}}^0$, $\sigma_{x_{\pm} p_{\pm}}^0$ and $\sigma_{p_{\pm} p_{\pm}}^0$ denote the variances at $t_0 = 0$. We note two missprints in [6]: $i)$ the presence of a global factor $\frac{1}{2}$ in the last term of $\sigma_{x_{\pm} p_{\pm}}$ and $ii)$ the last term of $\sigma_{x_{\pm} x_{\pm}}$ in [6] reads $2 h m \left( 2 f_{1, \pm} a_{11, \pm} + f_{1, \pm} a_{12, \pm} + a_{22, \pm} \right)$. Due to the discontinuity at $t_0 = 0$, variances at $t = 0^+$ jump to

$$\sigma_{x_{\pm} x_{\pm}}(t_0^+) = \sigma_{x_{\pm} x_{\pm}}^0, \quad \sigma_{x_{\pm} p_{\pm}}(t_0^+) = -\gamma \sigma_{x_{\pm} x_{\pm}}^0 + \sigma_{x_{\pm} p_{\pm}}^0, \quad \sigma_{p_{\pm} p_{\pm}}(t_0^+) = \gamma^2 \sigma_{x_{\pm} x_{\pm}}^0 - 2 \gamma \sigma_{x_{\pm} p_{\pm}}^0 + \sigma_{p_{\pm} p_{\pm}}^0,$$

where $t_0^+$ means $\lim t \to 0^+.$